PROPAGATION OF MAGNETOHYDRODYNAMIC SHOCK WAVES IN A MEDIUM OF DECREASING DENSITY

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It is explained under what condition instability develops in the wave front when a shock wave travels in a medium whose density is decreasing. It is shown that under laboratory conditions the buildup of such an instability may be suppressed by a diffusion of wave front segments into the walls of the system. Such an instability can occur, for example, in certain astrophysical bodies.

1. The front of a powerful shock wave traveling in a medium whose density decreases is unstable when there is no magnetic field present [1]. Random curving of the wave front, during which individual wave elements advance ahead of the front or fall behind it, becomes more frequent with time. Indeed, in a medium of decreasing density the front of a powerful shock wave moves at an increasing velocity. For this reason, a front element which has accidentally moved ahead will travel faster and its lead will become greater, while a front element which has accidentally fallen behind will travel slower and its lag will increase.

These qualitative concepts are also applicable to the case which will be considered here, namely, to the propagation of a strong shock wave (Mach number $M \gg 1$) in a medium of decreasing density in the presence of a transverse magnetic field. As has been shown in [2], the unperturbed front of a shock wave is accelerated when the Alfvén velocity $H_0(x)/\sqrt{4\pi\rho_0(x)}$ increases, $\rho_0(x)$ and $H_0(x)$ denoting here the unperturbed density of the medium and the magnetic field intensity, respectively. The subsequent calculation will validate these qualitative concepts also for the case of a magnetic shock wave.

2. We will consider small perturbations in the front of a wave whose length is much shorter than the length l of an inhomogeneity so that $kl \gg 1$, where k denotes the wave number of a perturbation, and so that the quasiclassical approximation applies. Furthermore, the medium will be considered perfectly conducting so that, by virtue of the "freeze-in" condition, the magnetic field intensity vector at every point on the front will be tangent to the front.

We will assume that the "curving" of the front is a function of the y-coordinate. We will designate an unperturbed front by X and the coordinate of a perturbed front by $\Xi = X + \xi$, with $\partial\xi/\partial y$ assumed small so that $(\partial\xi/\partial y)^2$ can be neglected. We then move the origin of the coordinate system to point Ξ and rotate the coordinates so that the new y'-axis will be tangent to the curved front. The angle of this rotation is equal to $\partial\xi/\partial y$ and is, therefore, small; with this accuracy, we have then

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} + \frac{\partial\xi}{\partial y}\frac{\partial}{\partial y'}, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'} - \frac{\partial\xi}{\partial y}\frac{\partial}{\partial x'} \tag{2.1}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \dot{\Xi}\frac{\partial}{\partial x'} - \dot{\Xi}\frac{\partial\xi}{\partial y}\frac{\partial}{\partial y'} - y'\frac{\partial^2\xi}{\partial y\partial t}\frac{\partial}{\partial x'} + x'\frac{\partial^2\xi}{\partial y\partial t}\frac{\partial}{\partial y'}$$

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3. In a laboratory system of coordinates the equations of magnetohydrodynamics at the front of a shock wave becomes

$$\begin{gathered}\frac{\partial \rho}{\partial t}+\frac{\partial \rho v_x}{\partial x}+\frac{\partial \rho v_y}{\partial y}=0 \\ \frac{\partial \rho v_x}{\partial t}+\frac{\partial}{\partial x}\left(p+\rho v_x^2+\frac{H^2}{8\pi}-\frac{H_x^2}{4\pi}\right)+\frac{\partial}{\partial y}\left(\rho v_x v_y-\frac{1}{4\pi} H_x H_y\right)=0 \\ \frac{\partial \rho v_y}{\partial t}+\frac{\partial}{\partial x}\left(\rho v_x v_y-\frac{1}{4\pi} H_x H_y\right)+\frac{\partial}{\partial y}\left(p+\rho v_y^2+\frac{H^2}{8\pi}-\frac{H_y^2}{4\pi}\right)=0 \\ \frac{\partial}{\partial t}\left(\frac{\rho v^2}{2}+\rho\varepsilon+\frac{H^2}{8\pi}\right)+\operatorname{div}\left\{\rho \mathbf{v}\left(\frac{v^2}{2}+w\right)+\frac{1}{4\pi}\left[\mathbf{H}\times(\mathbf{v}\times\mathbf{H})\right]\right\}=0\end{gathered} \tag{3.1}$$

where p, ρ, ε, and ω are the pressure, the density, the energy, and the enthalpy per unit mass.

Let us now turn to primed coordinates, then, integrate the equations with respect to x' over the jump interval, and with respect to y' over an infinitesimal region around the point of tangency between the y'-axis and the wave front. We must also consider that, within the established accuracy,

$$v_y = v_x \partial\xi/\partial y, \qquad H_x = H_y \partial\xi/\partial y$$

Then, for step increments in gasdynamic quantities, we obtain the following expressions:

$$\begin{gathered}\{\rho(v-\dot{\Xi})\}=0, \quad \left\{p+\rho v^2+\frac{H^2}{8\pi}-\rho v\dot{\Xi}\right\}=0 \\ \left\{\rho v\left(\frac{v^2}{2}+w+\frac{H^2}{4\pi\rho}\right)-\rho\dot{\Xi}\left(\frac{v^2}{2}+\varepsilon+\frac{H^2}{8\pi\rho}\right)\right\}=0\end{gathered} \tag{3.2}$$

After algebraic transformations based on the law of mass conservation, we have

$$\begin{gathered}\rho(v-\dot{\Xi})=-\rho_0\dot{\Xi}, \quad p^*=p_0^*+\rho_0 v\dot{\Xi} \\ (^1/_2 v^2+\varepsilon^*-\varepsilon_0^*)\,\rho_0\dot{\Xi}-p^* v=0 \\ p^*=p+H^2/8\pi, \quad \varepsilon^*=\varepsilon+H^2/8\pi\rho\end{gathered} \tag{3.3}$$

where the subscript "0" refers to the initial values of these quantities. To conditions (3.3), one must add the "freeze-in" condition (3.4):

$$H/\rho = H_0/\rho_0 \tag{3.4}$$

When perturbations occur, the values of functions ρ, v, p*, and H for a gas behind the wave front will differ from their unperturbed values by $\delta\rho$, δv, δp^*, and δH, respectively. If we also introduce the quantity $\delta u = \dot{\xi} - \xi\partial u/\partial x$, then, for an ideal gas, we obtain the following equations:

$$\begin{gathered}\left(\frac{u}{v}-1\right)\frac{\delta\rho}{\rho}-\frac{\delta v}{v}+\frac{\delta u}{u}=0 \\ \left(1-\frac{\rho c^*}{\rho_0 u}\right)\frac{\delta v}{v}+\frac{\delta u}{u}=0 \\ \left[\frac{H_0^2}{8\pi\rho_0^2}\rho-\frac{p}{(\gamma-1)\rho}\right]\frac{\delta\rho}{\rho}+\left[\frac{c^2 v}{(\gamma-1)c^*}-v^2-\frac{p_0^* v}{\rho_0 u}\right]\frac{\delta v}{v}+\frac{p_0^* v}{\rho_0 u}\frac{\delta u}{u}=0\end{gathered} \tag{3.5}$$

The following relations have been used here:

$$\begin{gathered}\varepsilon=p/(\gamma-1)\rho, \quad \delta p^*=\rho c^*\delta v \\ \delta p=(\rho c^2/c^*)\,\delta v \quad (c^*=\sqrt{\gamma p/\rho+H^2/4\pi\rho})\end{gathered}$$

with c_* denoting the velocity of sound, which become valid when terms of the order $(kl)^{-1}$ are disregarded [3].

The system of Eqs. (3.5) is homogeneous and its determinant is

$$\Delta=\frac{\lambda-\nu}{\lambda(\gamma-1)}\left[(1-\nu)(\gamma\nu-\lambda)-\gamma(\lambda-\nu)\left[\pi_0-\left(\frac{1}{\nu^2}-1\right)h_0^2\right]\right] \tag{3.6}$$

$$\nu=\frac{\rho_0}{\rho}, \quad h_0^2=\frac{H_0^2}{8\pi\rho_0 u^2}, \quad \pi_0=\frac{p_0}{\rho_0 u^2}, \quad \lambda=\frac{c^*}{u} \tag{3.7}$$

Using the results of [3], we also obtain

$$\nu = \frac{1}{2(\gamma+1)}[(\gamma-1)+2\gamma(\pi_0+h_0^2)+$$
$$+\sqrt{[(\gamma-1)+2\gamma(\pi_0+h_0^2)]^2+8(\gamma+1)(2-\gamma)h_0^2}] \tag{3.8}$$
$$\lambda = \sqrt{\gamma\nu\left[\pi_0-\left(\frac{1}{\nu^2}-1\right)h_0^2+(1-\nu)\right]+\frac{2h_0^2}{\nu}}$$

4. We will now consider special cases.

(1). There is no magnetic field present, $h_0 = 0$. If, also $\pi_0 = 0$ (a strong shock wave), then, $\Delta \neq 0$ except for $\gamma = 2$. Then, $\delta\rho = \delta v = \delta u = 0$, and

$$\frac{\partial\xi}{\partial t} = \xi\frac{\partial u}{\partial x}\Big|_{x=X} \tag{4.1}$$

If an unperturbed wave front is accelerated in the direction of decreasing density, then, the displacement $|\xi| \sim u$ increases, i.e., the displaced element separates from the unperturbed front. This occurs during perturbations of either sign.

(2). $\pi_0 = 0$, $0 < h_0^2 < \frac{1}{2}$ ($h_0^2 = \frac{1}{2}M^2$). An analysis will show that in this case, the determinant is $\Delta \neq 0$ everywhere except on the $h_0^2(\gamma)$ curve. For typical values of γ, we have the corresponding values of h_0^2:

γ	$^6/_5$	$^5/_4$	$^4/_3$	$^7/_5$	$^5/_3$	2
h_0^2	0.0278	0.0271	0.0254	0.0240	0.0152	0

When $\Delta \neq 0$, then, Eq. (4.1) is valid. In this case, when the Alfvén velocity $H_0(x)/\sqrt{4\pi\rho_0(x)}$ in an unperturbed gas increases in the direction of the front propagation, the velocity of the wave front increases and, consequently, displacements ξ will increase with the front becoming unstable.

When $\Delta = 0$, then, $\delta\rho/\rho$, $\delta v/v$, and $\delta u/u$ may approach unity. This means that the front of a shock wave is absolutely unstable, and any small front perturbations are becoming large.

5. If we let H_0 = const, then, the characteristic dimension of a perturbation buildup in a wave front will be αl, where $l = |\nabla \ln \rho_0|^{-1}$ is the so-called height of a homogeneous atmosphere, and α is a dimensionless coefficient (of the order of a few units) which can be found by numerical calculations. The characteristic time of instability buildup, according to Eq. (4.1), is of the order of the quantity $\alpha l/u$. During that time the wave front has broken down into separate small segments whose dimensions are of the order of l (height of a homogeneous atmosphere).

In the case of stellar catastrophes as, for example, flareups of novas or supernovas, this time is comparable to the time required for a shock wave associated with such an explosion to reach the star surface. Therefore, the wave front will appear very distorted when it reaches the surface, and it will not be spherical. This will considerably affect the luminance curve [4] and may explain the nature of the magnetic force lines in shedded shells. Indeed, the originally regular magnetic field of a star becomes interlinked with a strong shock wave appearing at the star surface. At the same time, it becomes amplified and distorted by it. If the magnetic field is sufficiently large, then, the development of small-scale pulsations may turn out to suppress the magnetic field accompanying large-scale pulsations. For this reason, the field structure is quasiregular in nature; its orientation in various large-scale elements is uncorrelated, but within each element, it has a preferred orientation. Just such peculiarities of the magnetic field can be observed in shells of flaring stars [5].

6. The propagation of a powerful shock wave in a medium of decreasing density is affected by energy cummulation processes, i.e., by an energy transfer from a large mass of a substance to a small mass. Such accumulation is particularly effective in the case of a magnetohydrodynamic wave (see, e.g., [6]). An instability of the kind analyzed here may occur in an appropriate laboratory experiment, when its buildup time is shorter than the diffusion time of front segments into the system walls, which, for example, is of the order of r/u or longer in the case of a shock wave traveling along a cylindrical tube of radius r. Therefore, the earlier stipulated condition is satisfied if $r > \alpha l$.

LITERATURE CITED

1. L. É. Gurevich and A. A. Rumyantsev, "Propagation of shock waves in a medium of decreasing density," Zh. Éksp. Teor. Fiz., 58, No. 4 (1970).
2. A. E. Voitenko and O. P. Sobolev, "Some cases of an accelerated magnetohydrodynamic shock wave," Zh. Prikl. Mekhan. i Tekh. Fiz., No. 2 (1968).
3. Pai Shih-I, Magnetogasdynamics and Plasma Dynamics, Springer-Verlag (1962).
4. L. É. Gurevich and A. A. Rumyantsev, "Theory of the luminance curve characterizing the flareup of supernovas," Astron. Zh., 47, No. 4 (1970).
5. I. S. Shklovskii, Supernovas [in Russian], Izd. Nauka, Moscow (1966).
6. A. E. Voitenko, M. A. Lyubimova, O. P. Sobolev, and V. S. Synakh, "Gradient acceleration of a shock wave," Preprint, Inst. Nuclear Physics, Novosibirsk (1970).